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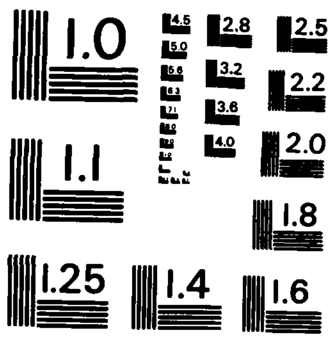

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TECHNICAL REPORT ARLCB-TR-85027

DEFLECTION IN TAPERED CANTILEVER BEAMS DEFLECTION (GAP OPENING) IN DOUBLE CANTILEVER TYPE FRACTURE TOUGHNESS SPECIMENS

BOAZ AVITZUR

AUGUST 1985

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT CENTER
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When an otherwise homogeneous material under stress contains small defects (i.e., internal cracks and/or voids), the stresses at parts of the material-defect interface significantly exceed the ones anticipated at that location in the absence of such irregularities. Consequently, a structural member, otherwise calculated to safely sustain the applied loads, might unpredictably fail. That branch of engineering which intends to account for such 'stress-raisers' is known as fracture mechanics. Fracture mechanics studies have found that

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20. ABSTRACT (CONT'D)

different materials (and even the same material when loaded in different orientations) reflect different sensitivity to such 'stress-raisers'--a material property known as fracture toughness. Test samples and testing procedures have been devised in order to quantify this material property. The relation between the applied load and its displacement (or gap opening) at the point of crack growth is being used herein to determine (compute) material fracture toughness.

While the equations derived for the stress field near the edge of a defect in an otherwise uniform field assume an infinite volume of material to surround the (relatively) very small defect, the crack to width and/or height in these laboratory size testing samples is definitely a finite one. This report offers a mathematical relation between the applied load and that part of the deflection (gap opening) which is due to the cantilever-like part of the sample, for that class of fracture toughness test specimens which can be described as double cantilever. A beam theory approach is used.

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INTRODUCTION

One type of specimen commonly used in tests designed for the determination of a material's fracture toughness can be looked upon as a double cantilever beam, as shown in Figures 1 and 2.

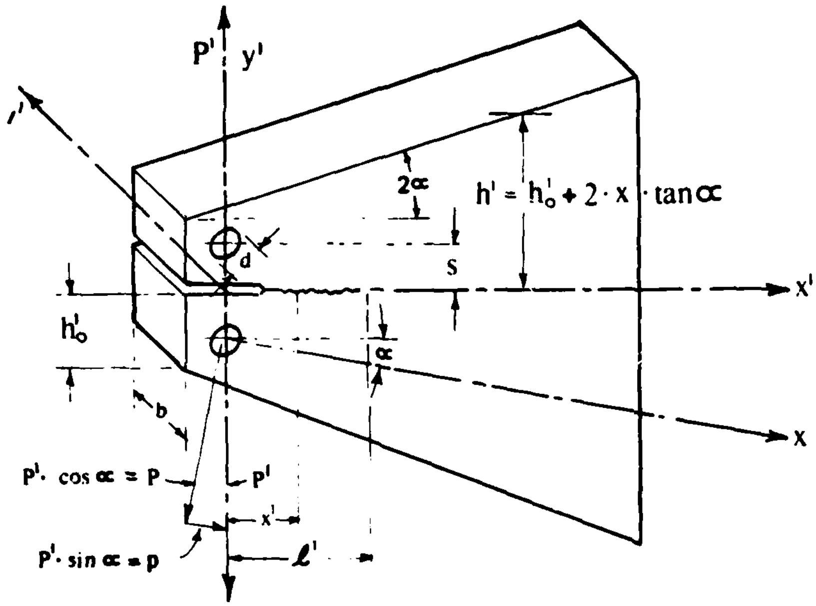

Figure 1. Tapered double cantilever beam.

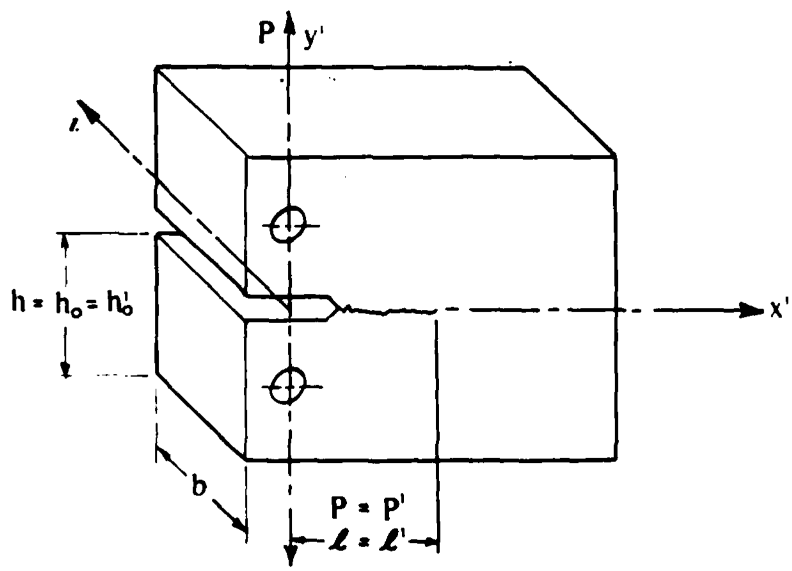

Figure 2. Compact fracture toughness specimen.

The data collected in such tests to determine the material's fracture toughness property includes the gap opening, $2\delta'$, at a given load, P', and for a given crack length, ℓ'. The stress distribution in such specimens is a complex one and so is the resultant strain distribution and its contribution to the gap opening. The computation of that portion of the gap opening attributed to the elastic bending of a cantilever beam is the subject of this report. These calculations are based on an elastic beam theory. The shortcomings of the computed gap opening derived are as follows:

1. The fracture rarely propagates without attaining a plastically deformed region near the tip of the crack.

2. The calculations do not account for the deformation in the remaining portion of the specimen, namely at the areas $x' > \ell'$, beyond the crack length.

The added gap opening due to the propagation of plastic deformation should be computed separately using the technique used by this author elsewhere (ref 1). The contribution of the deformation in the region $x' > \ell'$ towards the gap opening, can also be computed separately. (Neither one of these calculations is included here.)

In treating each half of the specimen as a tapered cantilever, one has to realize that its neutral axis in bending is not parallel to the specimen's plane of symmetry (x' axis in Figure 1), but rather at an angle α to it (as will be shown later, the effective neutral plane deviates from the bisecting plane of each cantilever, too). As a consequence, the computation of the

[1]Boaz Avitzur, "Retained Deflection in Circular and Concentrically Hollowed Beams After Local Removal," to be published.

bending moment and of the shear forces, as well as that of the resultant stresses thereof, will be based on the component $P = P' \cdot \cos\alpha$ of the applied load P', where P acts normal to the axis of symmetry, x (where $x = [x' - \frac{h_o}{2} \tan \alpha] \cos \alpha$).

PARALLEL STRESS COMPONENTS

As a first approximation, the gap opening $\delta = \delta_{bend} + \delta_{shear}$ is calculated in the cartesian coordinates system x-y-z, which is at an angle α to the x'-y'-z' coordinates and only the normal component, σ_{xx}, and the shear component, $\sigma_{xy} = \sigma_{yx}$, are considered. This is followed later by correcting the stress components to be parallel to the external surfaces at the boundaries

$$y = \pm \frac{h_o + 2x' \tan \alpha}{2}$$

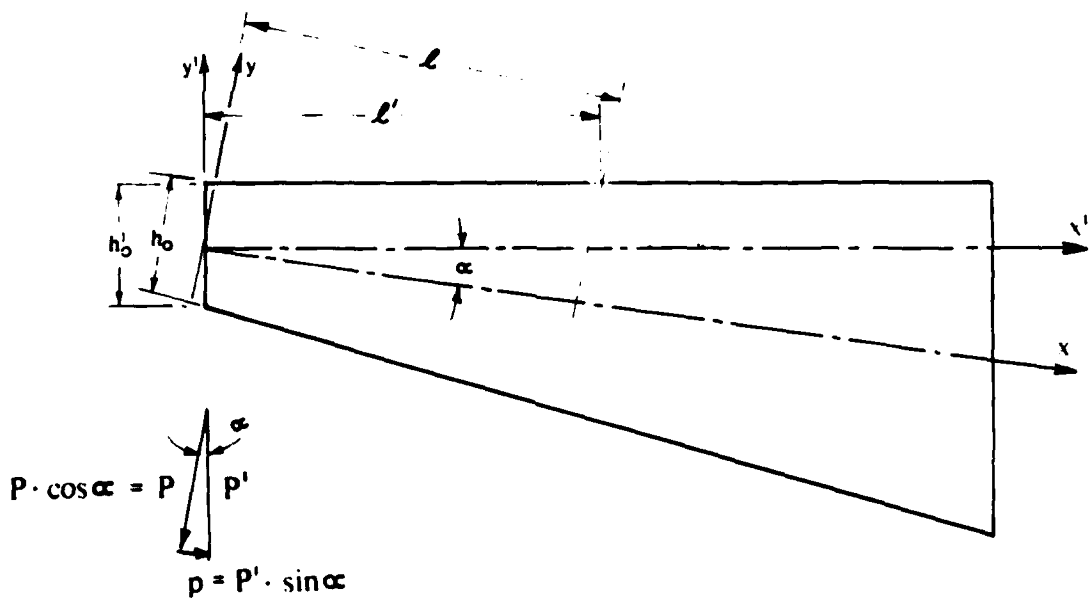

Figure 3. Definition of cartesian coordinates and major parameters.

The basic dimensions in the x-y coordinates and their relation to those in the x'-y' coordinates (see Figure 3) are as follows:

$h_o = \frac{h_o'}{\cos \alpha}$ ≡ the beam's cross-sectional height at the point of load application

$x = (x' - \frac{h_o}{2} \tan \alpha) \cdot \cos \alpha$, where x' is being measured along the top surface of the beam, which is the fracture toughness specimen, along the crack surface.

$\ell = (\ell' - \frac{h_o'}{2} \tan \alpha) \cdot \cos \alpha$ ≡ distance of crack tip from point of load application

$t = 2 \cdot \tan \alpha$

$P = P' \cdot \cos \alpha$ ≡ component of load normal to plane of symmetry

$p = P' \cdot \sin \alpha$ ≡ component of load parallel to plane of symmetry

$h = h_o + t \cdot x$ ≡ beam's height normal to the plane of symmetry at a distance x from the point of load application.

Thus, the cross-sectional moment of inertia becomes

$$I = \frac{bh^3}{12} = b\,\frac{(h_o+tx)^3}{12} \tag{1}$$

and the local curvature ρ of the beam is

$$\frac{1}{\rho} = \frac{M(x)}{E \cdot I(x)} = \frac{Px}{E \cdot I(x)} = 12\,\frac{P}{Eb} \cdot \frac{x}{(h_o+tx)^3} \tag{2}$$

and the total deflection due to bending moment is

$$y = y_o + \int_o^x \theta_o dx + 12\,\frac{P}{Eb} \int_o^x \int_o^x \frac{x}{(h_o + tx)^3} \cdot dx \cdot dx \tag{3}$$

for which

$$\theta = \theta_o + 12 \frac{P}{Eb} \int_o^x \frac{x}{(h_o+tx)^3} \cdot dx = \theta_o + 12 \frac{P}{Ebt^2} \left[- \frac{1}{h_o + tx} + \frac{h_o}{2(h_o+tx)^2}\right]_o^x =$$

$$\theta_o - 12 \frac{P}{Eb} \cdot \frac{x^2}{2h_o(h_o+tx)^2} \qquad (4)$$

where θ = angle of deflection

θ_o = angle of deflection at the point of load application.

However, at the tip of the crack, $x = \ell$, $\theta_\ell = 0$. Therefore,

$$\theta_\ell = 0 = \theta_o - 12 \frac{P}{Eb} \cdot \frac{\ell^2}{2h_o(h_o+t\ell)^2}$$

from which we can determine that

$$\theta_o = 6 \frac{P}{Eb} \cdot \frac{\ell^2}{h_o(h_o+t\ell)^2} \qquad (5)$$

By applying Eq. (5) to Eq. (3), one gets

$$y = y_o + \int_o^x \theta_o dx + 12 \frac{P}{Eb} \int_o^x \int_o^x \frac{x}{(h_o+tx)^3} dxdx$$

$$= y_o + 6 \frac{P}{Ebh_o} \left\{\frac{\ell^2}{(h_o+t\ell)^2} \int_o^x dx - \int_o^x \frac{x^2}{(h_o+tx)^2} dx\right\}$$

$$= y_o + 6 \frac{P}{Ebh_o} \left\{\frac{\ell^2}{(h_o+t\ell^2)} \Big/_o^x - \frac{1}{t^3} \left[h_o+tx - 2h_o \log(h_o+t_x) - \frac{h_o}{h_o + tx}\right]_o^x\right\}$$

$$= y_o + 6 \frac{P}{Ebh_o} \left\{\frac{\ell^2}{(h_o+t\ell)^2} x - \frac{1}{t^3} \left[\frac{2h_o + tx}{h_o + tx} \cdot tx - 2h_o \log \frac{h_o + tx}{h_o}\right]\right\} \qquad (6)$$

Since at the tip of the crack, $x = \ell$, and the deflection $y_\ell = 0$, one gets

$$y_\ell = 0 = y_o + 6 \frac{P}{Ebh_o} \left\{ \frac{\ell^3}{(h_o+t\ell)^2} - \frac{1}{t^3} \left[\frac{2h_o + t\ell}{h_o + t\ell} t\ell - 2h_o \log \frac{h_o + t\ell}{h_o} \right] \right\} \quad (7)$$

However, for a straight cantilever where $t = 0$, one has to determine

$$y_\ell = 0 = y_o + 6 \frac{\ell^3}{Ebh_o} \left\{ \frac{\ell^3}{(h_o+t\ell)^2} - \lim_{t \to 0} \left(\frac{1}{t^3} \left[\frac{2h_o + tx}{h_o + tx} t_x - 2h_o \log \frac{h_o + tx}{h_o} \right] \right) \right\}$$

where

$$\lim_{t \to 0} \left\{ \frac{1}{t^3} \left[\frac{2h_o + tx}{h_o + tx} tx - 2h_o \log \frac{h_o + tx}{h_o} \right] \right\} = \lim_{t \to 0} \frac{x^3}{3(h_o+tx)^2} = \frac{x^3}{3h_o^2} \quad (8)$$

from which

$$\lim_{t \to 0} y_o = -6 \frac{P}{Ebh_o} \left\{ \frac{\ell^3}{h_o^2} - \frac{\ell^3}{3h_o^2} \right\} = -4 \frac{P}{Eb} \left(\frac{\ell}{h_o} \right)^3 = y_o' \quad (9)$$

which is the same as being reported in handbooks (ref 2) for a straight cantilever of a uniform cross-section.

Another method of computing the gap opening is by invoking Castigliano's theorem (ref 3), equating the internal deformation energy with that of the outside work done on the structural member. This method will facilitate computation of the gap opening due to bending moment and due to shear force (and any other forces that impose internal deformation work).

The distribution of that component of normal stresses, $\sigma_{xx,m}$, which is parallel to the axis of symmetry and which results from the bending moment, is as follows:

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[2]Raymond J. Roark and Warren C. Young, Formulas for Stress and Strain, Fifth Edition, McGraw-Hill, NY, 1975, p. 98.

[3]A. C. Ugural and S. K. Fenster, Advanced Strength and Applied Elasticity, American Elsevier, NY, 1981, pp. 146-148.

$$\sigma_{xx,m} = \frac{M}{I} \cdot y = \frac{P}{I} \cdot x \cdot y \tag{10}$$

where $P = P' \cos \alpha$, $\alpha = \arctan (t/2) = \arctan (\Delta h/2\Delta x)$, and $I = bh^3/12 = \frac{b}{12} \cdot (h_o+tx)^3$.

The component of shear stress which lies in a plane normal to the plane of symmetry and in the direction of the axis of symmetry (or opposite to it), $\sigma_{yx,m}$ and its equivalent $\sigma_{xy,m}$, is assumed to be due to gradient in that component of the normal stress $d\sigma_{xx,m}/dx$, resulting from the bending moment. This is computed as follows:

$$\sigma_{yx,m} = \frac{2 \int_o^{h/2} \int_o^{b/2} \frac{\Delta\sigma_{xx,m}}{\Delta x} dzdy}{2}$$

where

$$\sigma_{xx,m} = \frac{M}{I} \cdot y$$

Thus

$$\frac{d\sigma_{xx,m}}{dx} = y \frac{d}{dx} (\frac{M}{I}) = P.y \cdot \frac{d}{dx} (\frac{12}{b(h_o+tx)^3} \cdot x) = 12 \frac{P}{b} y \frac{d}{dx} (\frac{x}{(h_o+tx)^3})$$

where

$$\frac{d}{dx} (\frac{x}{(h_o+tx)^3}) = \frac{h_o + tx - 3tx}{(h_o+tx)^4} = \frac{h_o - 2tx}{(h_o+tx)^4} = \frac{1}{(h_o+tx)^3}[1 - 3 \frac{tx}{h_o + tx}]$$

and thus

$$\sigma_{yx,m} = \frac{24 \frac{P}{b} \int_y^{(h_o+tx)/2} \int_o^{b/2} \frac{h_o - 2tx}{(h_o+tx)^4} y \cdot dz \cdot dy}{b} = \frac{12P}{b} \int_y^{(h_o+tx)/2} \frac{h_o - 2tx}{(h_o+tx)^4} \cdot y \cdot dy$$

$$\sigma_{yx,m} = \frac{3P}{2b} \cdot \frac{h_o - 2tx}{(h_o+tx)^4} [(h_o+tx)^2 - 4y^2] = \frac{3P}{2b} \cdot \frac{h_o - 2tx}{(h_o+tx)^2} [1 - (2 \frac{y}{h_o+tx})^2]$$

$$= \frac{3P'}{2b} \cdot \frac{h_o - 2tx}{(h_o+tx)^2} [1 - (2 \frac{y}{h_o+tx})^2] \cdot \cos \alpha \tag{11}$$

Thus

$$U_{bend3} = (12 + \frac{9}{5} t^2) \frac{P^2}{bE} \int_o^x \frac{x^2}{(h_o+tx)^3} dx$$

where

$$\int_o^x \frac{x^2}{(h_o+tx)^3} dx = \frac{1}{t^3} [\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx]$$

Thus

$$U_{bend3} = (\frac{12}{t^3} + \frac{9}{5} \frac{1}{t}) \frac{P^2}{bE} [\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^3} tx] = P\delta_{bend3}$$

from which

$$\delta_{bend3} = (\frac{12}{t^3} + \frac{9}{5} \frac{1}{t}) \frac{P}{bE} [\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx] \qquad (26)$$

and as before

$$\delta'_{bend3} = (\frac{12}{t^3} + \frac{9}{5} \frac{1}{t}) \frac{P}{bE} [\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx]\cdot\cos^2 \alpha \qquad (26a)$$

At the limit, as $t \to 0$, for a beam of a constant cross-section,

$$\lim_{t\to 0} \delta_{bend3} = \frac{P}{bE} \lim_{t\to 0} \{(\frac{12}{t^3} + \frac{9}{5} \frac{1}{t})[\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx]\}$$

where

$$\lim_{t\to 0} \{\frac{1}{t^3} [\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx]\} = \lim_{t\to 0} \frac{\frac{d}{dt} [\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx]}{\frac{d}{db} (t^3)}$$

$$= \lim_{t\to 0} \frac{\frac{x}{h_o + tx} - \frac{[(2h_o+3tx)x + 3tx^2](h_o+tx)^2 - 2(2h_o+3tx)\cdot tx\cdot(h_o+tx)\cdot x}{2(h_o+tx)^4}}{3t^2}$$

Since $\sigma_{xx,m}$ (or $\sigma_{rr,m}$) is tensile in the range $0 < \theta < \alpha$ (or $0 < y < (h_o+tx)/2$) and compressive in the range $-\alpha < \theta < 0$ (or $-(h_o+tx)/2 < y < 0$), $\sigma_{yy,m} = \sigma_{xx,m}\cdot\tan\theta$ maintains its upwards component and these do not cancel each other (Figure 5). Nevertheless, no discontinuity prevails at $\theta = 0$ (or $y = 0$), since $\sigma_{yy,m} = 0$ on that (internal) surface. The contribution of the stress component, $\sigma_{yy,m}$ to the deflection is accounted for by the energy balance in its totality.

Deflection Due to Bending Forces Only

Since

$$\sigma_{xx,m} = 12\,\frac{P}{b}\,\frac{xy}{(h_o+tx)^3}$$

$$\sigma_{rr,m} = 12\,\frac{P}{b}\sqrt{1 + \frac{t^2y^2}{(h_o+tx)^2}}\,\frac{xy}{(h_o+tx)^3}$$

from which the local strain energy, due to bending moment, u_{bend3} becomes

$$u_{bend3} = \sigma_{rr,m}\cdot\epsilon_{rr,m} = \frac{\sigma_{rr,m}^2}{E} = 144\,\frac{P^2}{b^2E}\left[1 + \frac{t^2y^2}{(h_o+tx)^2}\right]\frac{x^2y^2}{(h_o+tx)^6}$$

Hence, one can compute the total strain energy due to the bending moment as follows:

$$U_{bend3} = \int_v u_{bend3}\cdot dv = 576\,\frac{P^2}{b^2E}\int_o^x\int_o^{(h_o+tx)/2}\int_o^{b/2}\left[1 + \frac{x^2y^2}{(h_o+tx)^2}\right]\frac{x^2y^2}{(h_o+tx)^6}\cdot dz\cdot dy\cdot dx$$

$$= 288\,\frac{P^2}{bE}\int_o^x\int_o^{(h_o+tx)/2}\left[1 + \frac{t^2y^2}{(h_o+tx)^2}\right]\frac{x^2y^2}{(h_o+tx)^6}\cdot dy\cdot dx$$

where

$$\int_o^{(h_o+tx)/2}\left[1 + \frac{t^2y^2}{(h_o+tx)^2}\right]\cdot\frac{x^2y^2}{(h_o+tx)^6}\,dy = \frac{x^2}{(h_o+tx)^6}\left[\frac{y^3}{3} + \frac{t^2}{(h_o+tx)^2}\,\frac{y^5}{5}\right]_o^{(h_o+tx)/2}$$

$$= \frac{x^2}{(h_o+tx)^3}\left[\frac{1}{24} + \frac{t^2}{160}\right]$$

of any cross-section at a distance x from the line of load application is $L = L'+x = \frac{h_o}{t} + x$ from the origin of these cylindrical coordinates (where $t = 2\cdot\tan\alpha$), and the distance ℓ of any point (x,y) on a plane x and at a distance y from the axis of symmetry is:

$$\ell = \sqrt{L^2 + y^2} = \sqrt{\left(\frac{h_o}{t} + x\right)^2 + y^2} = \frac{1}{t}\sqrt{(h_o+tx)^2 + t^2y^2}$$

from which

$$\frac{\sigma_{rr}}{\sigma_{xx}} = \frac{\ell}{L} = \frac{\sqrt{(h_o+tx)^2 + t^2y^2}}{h_o + tx} = \sqrt{1 + \left(\frac{ty}{h_o+tx}\right)^2}$$

or

$$\sigma_{rr}^2 = \left[1 + \frac{t^2y^2}{(h_o+tx)^2}\right]\sigma_{xx}^2$$

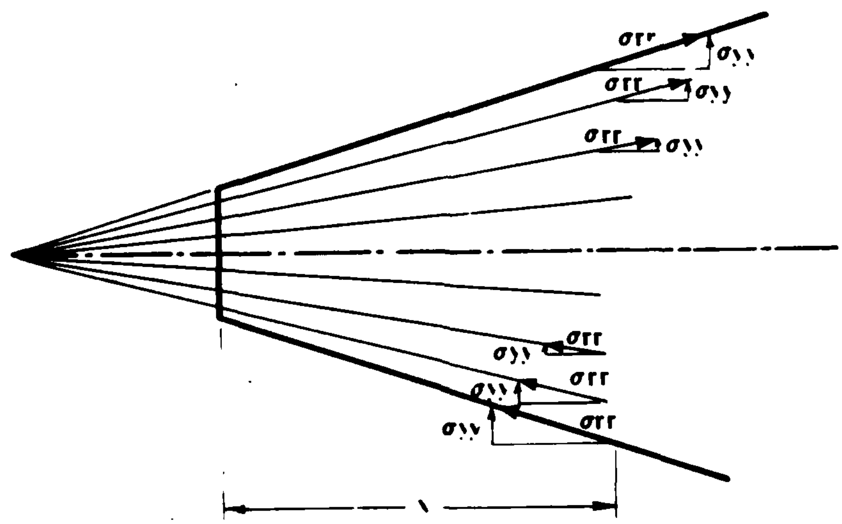

Figure 5. Radial and cartesian components of normal stresses.

CONVERGING STRESS COMPONENTS

By assuming that the normal stresses, σ_{xx}, are parallel to the plane of symmetry, one violates the boundary condition at $y = \pm (h_o+tx)/2$, in a tapered beam. Therefore, it will be assumed here that the normal stresses, σ_{rr}, at the surfaces of $y = \pm (h_o+tx)/2$ (or $\theta = \pm \alpha$) (see Figure 4) are parallel to these surfaces and that the radial stresses gradually align themselves with the axis of symmetry ($\theta = 0$). Or, in other words, these are radial stresses, as their subscript suggests, in a cylindrical coordinate with the origin at $L' = (h_o/2)\cdot\tan\alpha = h_o/t$ distance from the point where $x = 0$. Thus, the distance L

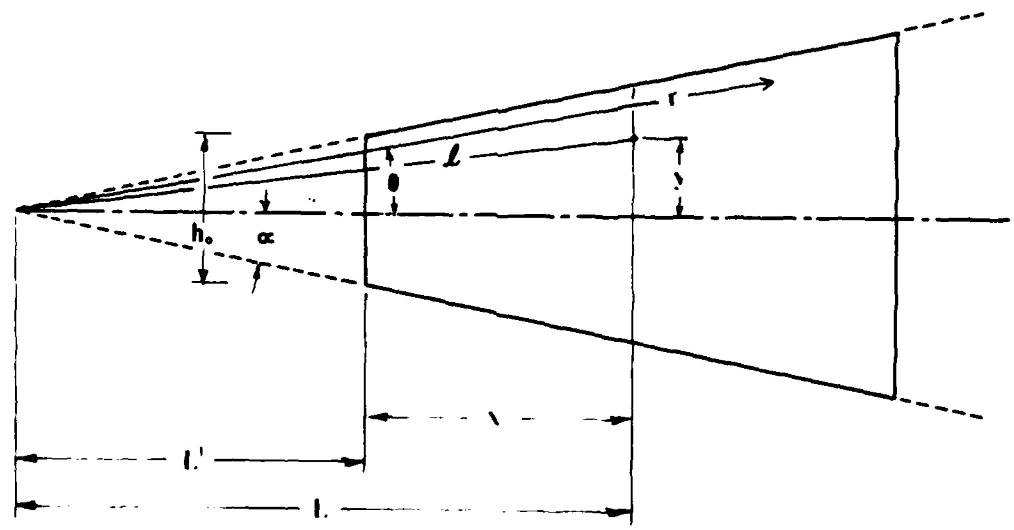

Figure 4. Conversion from cartesian to cylindrical coordinate systems.

Hence

$$\delta_{shear2} = \frac{1+\nu}{E}\,\frac{P}{b}\,\{\frac{12}{5}\,\frac{1}{t}\,[4\cdot\log\frac{h_o + tx}{h_o} - 3\cdot\frac{2h_o + 5tx}{2(h_o+tx)^2}\,tx] + \frac{t^3}{24}\cdot\log\frac{h_o + tx}{h_o}\} \quad (24)$$

and

$$\delta'_{shear2} = \frac{1+\nu}{E}\,\frac{P'}{b}\{\frac{12}{5}\cdot\frac{1}{t}\cdot[4\cdot\log\frac{h_o + tx}{h_o} - 3\,\frac{2h_o + 5tx}{2(h_o+tx)^2}\,tx] + \frac{t^3}{24}\,\log\frac{h_o + tx}{h_o}\}\cos^2\alpha \quad (24a)$$

Combining Eqs. (22) and (24), one gets

$$\frac{\delta_{shear2}}{\delta_{bend2}} = \frac{\frac{1+\nu}{E}\,\frac{P}{b}\,\{\frac{12}{5}\,\frac{1}{t}\,[4\cdot\log\frac{h_o + tx}{h_o} - 3\,\frac{2h_o + 5tx}{2(h_o+tx)^2}\,tx] + \frac{t^3}{24}\,\log\frac{h_o + tx}{h_o}\}}{\frac{P}{bE}\,\{\frac{12}{t^3}\,[\log\frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2}\,tx] + \frac{t}{4}\,\log\frac{h_o + tx}{h_o}\}} \quad (25)$$

As before (in computing $\lim_{t\to 0} \delta_{bend}$),

$$\lim_{t\to 0}\,\{\frac{t^3}{24}\,\log\frac{h_o + tx}{h_o}\} = 0$$

Therefore

$$\lim_{t\to 0}\,\delta_{shear2} = \frac{12}{5}\,\frac{P}{b}\,\frac{1+\nu}{E}\,(\frac{x}{h_o})$$

as in Eq. (16), and

$$\lim_{t\to 0}\,\frac{\delta_{shear2}}{\delta_{bend2}} = \frac{3}{5}\,(1+\nu)(\frac{h_o}{x})^2$$

as in Eq. (17).

Thus, the total shear strain energy due to the combined bend and compressive forces and over the entire volume of the beam U_{shear2} becomes

$$U_{shear2} = 2b \int_o^x \int_o^{(h_o+tx)/2} u_{shear2} \cdot dy \cdot dx = \frac{1+\nu}{E} \frac{P^2}{b} \int_o^x \int_o^{(h_o+tx)/2} 9(h_o{}^2 4h_o tx + 4t^2x^2) \cdot$$

$$\left[\frac{1}{(h_o+tx)^4} - \frac{8y^2}{(h_o+tx)^6} + \frac{16y^4}{(h_o+tx)^8}\right]$$

$$+ \frac{t^4}{4} \left[\frac{1}{(h_o+tx)^2} - \frac{4|y|}{(h_o+tx)^3} + \frac{4y^2}{(h_o+tx)^4}\right] \}dydx$$

As in the derivation of Eq. (4)

$$\int_o^{(h_o+tx)/2} \left[\frac{1}{(h_o+tx)^4} - \frac{8y^2}{(h_o+tx)^6} + \frac{16y^2}{(h_o+tx)^8}\right] dy = \frac{4}{15} \cdot \frac{1}{(h_o+tx)^3}$$

and

$$\int_o^{(h_o+tx)/2} \left[\frac{1}{(h_o+tx)^2} - \frac{4|y|}{(h_o+tx)^3} + \frac{4y^2}{(h_o+tx)^4}\right] dy = \left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right] \cdot \frac{1}{h_o + tx} = \frac{1}{6(h_o+tx)}$$

Thus

$$U_{shear2} = \frac{1+\nu}{E} \frac{P^2}{b} \cdot \int_o^x \left\{\frac{12}{5} \frac{h_o{}^2 - 4h_o tx + 4t^2x^2}{(h_o+tx)^3} - \frac{t^4}{24} \cdot \frac{1}{h_o + tx}\right\} dx$$

As in the derivation of Eq. (14)

$$\int_o^x \frac{h_o{}^2 - 4h_o tx + 4t^2x^2}{(h_o+tx)^3} dx = \frac{1}{t} \left\{4 \cdot \log \frac{h_o + tx}{h_o} - 3 \frac{2h_o + 5tx}{2(h_o+tx)^2} tx\right\}$$

whereas

$$\int_o^x \frac{dx}{h_o + tx} = \frac{1}{t} \log \frac{h_o + tx}{h_o}$$

Thus

$$U_{shear2} = \frac{1+\nu}{E} \frac{P^2}{b} \left\{\frac{12}{5} \cdot \frac{1}{t} \cdot \left[4 \cdot \log \frac{h_o + tx}{h_o} - 3 \frac{2h_o + 5tx}{2(h_o+tx)^2} tx\right] + \frac{t^3}{24} \cdot \log \frac{h_o + tx}{h_o}\right\}$$

$$= P\delta_{shear2}$$

as in Eq. (9), since

$$\lim_{t \to 0} \{\frac{t}{4} \log \frac{h_o + tx}{h_o}\} = 0$$

By adding the two components of shear stresses due to the bend forces, P' cos α, and due to the compressive forces, P' sin α, from Eqs. (11) and (18), respectively, one gets

$$\sigma_{yx} = \sigma_{yx,m} + \sigma_{yx,p} = \frac{P}{2b} \cdot \{3 \frac{h_o - 2tx}{(h_o+tx)^2} \cdot [1 - (2 \frac{y}{h_o+tx})^2] - \frac{t^2}{2(h_o+tx)} \cdot [1 - 2 \frac{|y|}{h_o+tx}]\}$$

Thus, the local combined shear strain energy, u_{shear}, due to the bending and compressive forces is

$$u_{shear2} = \sigma_{yx}\varepsilon_{yx} = 2 \frac{1+\nu}{E} \sigma_{yx}{}^2 = \frac{1+\nu}{E} \frac{P^2}{2b^2} \{3 \frac{h_o - 2tx}{(h_o+tx)^2} \cdot [1 - (2 \frac{y}{h_o+tx})^2]$$

$$- \frac{t^2}{2(h_o+tx)} \cdot [1 - 2 \frac{|y|}{h_o + tx}]\}^2$$

$$= \frac{1+\nu}{E} \frac{P^2}{2b^2} \cdot \{9 \frac{h_o{}^2 - 4h_o tx + 4t^2x^2}{(h_o+tx)^4} \cdot [1 - 2(2 \frac{y}{h_o+tx})^2 + (2 \frac{y}{h_o+tx})^4]$$

$$- 3 \cdot \frac{h_o - 2tx}{(h_o+tx)^3} \cdot [1 - (2 \frac{y}{h_o+tx})^2] \cdot [1 - 2 \frac{|y|}{h_o+tx}] \cdot t^2 + \frac{t^4}{4(h_o+tx)^2} \cdot$$

$$[1 - 4 \frac{|y|}{h_o + tx} + 4 \frac{y^2}{(h_o+tx)^2}]\} \qquad (23)$$

where, due to symmetry around the x-axis, the term

$$3 \frac{h_o - 2tx}{(h_o+tx)^3} \cdot [1 - (2 \frac{y}{h_o+tx})^2] \cdot [1 - 2 \frac{|y|}{h_o + tx}] \cdot t^2$$

cancels itself on both sides of the axis of symmetry.

$$U_{bend2} = 2b \int_0^x \int_0^{(h_o+tx)/2} u_{bend2} \cdot dy \cdot dx = \frac{P^2}{2bE} \int_0^x \int_0^{(h_o+tx)/2} \{576 \frac{x^2 y^2}{(h_o+tx)^4} + t^2\} \frac{1}{(h_o+tx)^2} \cdot dy \cdot dx$$

where, as in the derivation of Eq. (12)

$$\int_0^{(h_o+tx)/2} \frac{x^2 y^2}{(h_o+tx)^6} dy = \frac{1}{24} \frac{x^2}{(h_o+tx)^3}$$

In addition,

$$\int_0^{(h_o+tx)/2} \frac{t^2}{(h_o+tx)^2} dy = \frac{t^2}{2(h_o+tx)}$$

Thus

$$U_{bend2} = \frac{P^2}{2bE} \int_0^x \{24 \frac{x^2}{(h_o+tx)^3} + \frac{t^2}{2(h_o+tx)}\} dx$$

$$= \frac{P^2}{2bE} \{\frac{24}{t^3} [\log(h_o+tx) + \frac{2h_o}{h_o + tx} - \frac{h_o{}^2}{2(h_o+tx)^2}] + \frac{t}{2} \log(h_o+tx)\}_0^x$$

$$= \frac{P^2}{bE} \{\frac{12}{t^3} [\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx] + \frac{t}{4} \log \frac{h_o + tx}{h_o}\} = P\delta_{bend2} \qquad (21)$$

from which

$$\delta_{bend2} = \frac{P}{bE} \{\frac{12}{t^3} [\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx] + \frac{t}{4} \log \frac{h_o + tx}{h_o}\} \qquad (22)$$

and as before

$$\delta'_{bend2} = \frac{P'}{bE} \{\frac{12}{t^3} [\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx] + \frac{t}{4} \log \frac{h_o + tx}{h_o}\} \cos^2 \alpha \qquad (22a)$$

which, as was previously shown (following the derivation of Eq. (12))

$$\lim_{t \to 0} \delta_{bend2} = -4 \frac{P}{bE} (\frac{x}{h_o})^3$$

Considering the symmetry of the beam, one arrives at

$$\sigma_{yx,p} = \frac{-Pt^2}{4b(h_o+tx)} [1 - 2 \frac{|y|}{h_o + tx}] = \frac{-P't}{4b(h_o+tx)} [1 - 2 \frac{|y|}{h_o + tx}] \cdot \sin \alpha \quad (18a)$$

By adding the two components of normal stresses due to the bend forces, $P' \cdot \cos \alpha$, and due to the compressive forces, $P' \sin \alpha$, one gets

$$\sigma_{xx} = \sigma_{xx,m} + \sigma_{xx,p} = 12 \frac{P}{b} \cdot \frac{xy}{(h_o+tx)^3} - \frac{P}{2b} \cdot \frac{t}{h_o + tx} = \frac{P}{2b} \{24 \frac{xy}{(h_o+tx)^2} - t\} \frac{1}{h_o + tx} \quad (19)$$

The term

$$\{24 \frac{xy}{(h_o+tx)^2} - t\}$$

in Eq. (19) is not symmetric in respect to the plane of symmetry ($y = 0$) (except for beams of constant cross-section, where $t = 0$). Thus the beam's geometrical plane of symmetry is not necessarily the plane of symmetry for the stress distribution.

As in the derivation of Eq. (12), the local strain energy due to normal stresses, u_{bend2}, is

$$u_{bend2} = \sigma_{xx}\epsilon_{xx} = \frac{\sigma_{xx}{}^2}{E} = \frac{P^2}{4b^2E} \{576 \frac{x^2y^2}{(h_o+tx)^4} - 48 \frac{xy}{(h_o+tx)^2} t + t^2\} \frac{1}{(h_o+tx)^2} \quad (20)$$

However, in integrating over the entire volume and due to the symmetry around the x-axis, the form

$$48 \frac{xy}{(h_o+tx)^2} \cdot t$$

cancels itself on both sides of this axis. Thus, the total strain energy due to the normal stresses and over the entire volume, U_{bend}, is

Thus

$$\lim_{t\to 0} \delta_{shear1} = \frac{12}{5}\frac{P}{6}\frac{1+\nu}{E}(4-3)\left(\frac{x}{h_o}\right) = \frac{12}{5}\frac{P}{b}\frac{1+\nu}{E}\left(\frac{x}{h_o}\right) \qquad (16)$$

and together with Eq. (9) (where $y_o = \delta_{bend}$)

$$\lim_{t\to 0} \frac{\delta_{shear1}}{\delta_{bend1}} = \frac{\frac{12}{5}\frac{P}{b}\frac{1+\nu}{E}\left(\frac{x}{h_o}\right)}{4\frac{P}{bE}\left(\frac{x}{h_o}\right)^3} = \frac{3}{5}(1+\nu)\left(\frac{h_o}{x}\right)^2 \qquad (17)$$

Deflection Due to Combined Bending and Compressive Forces

If one incorporates the effect of the compressive stresses

$$\sigma_{xx,p} = \frac{P'}{b(h_o+tx)}\sin\alpha = \frac{P}{b(h_o+tx)}\frac{\sin\alpha}{\cos\alpha} = \frac{Pt}{2b(h_o+tx)}$$

then the change in compressive stresses due to change in cross-sectional area becomes

$$\frac{d\sigma_{xx,p}}{dx} = -\frac{P}{2b}\frac{t^2}{(h_o+tx)^2} = -\frac{P'}{b}\frac{t}{(h_o+tx)^2}\sin\alpha$$

from which the associated shear stresses will be

$$\sigma_{yx,p} = \frac{2\int_y^{(h_o+tx)/2}\int_o^{b/2}\frac{d\sigma_{xx,p}}{dx}\cdot dz\cdot dy}{b} = \frac{-2\frac{Pt^2}{2b(h_o+tx)^2}\int_y^{(h_o+tx)/2}\int_o^{b/2}dz\cdot dy}{b}$$

$$= \frac{-Pt^2}{2b(h_o+tx)^2}\left[\frac{h_o+tx}{2} - y\right] = \frac{-Pt^2}{4b(h_o+tx)}\left][1 - 2\frac{y}{h_o+tx}\right]$$

$$= \frac{-P't}{4b(h_o+tx)}\left[1 - 2\frac{y}{h_o+tx}\right]\cdot\sin\alpha \qquad (18)$$

and as before

$$\delta'_{shear1} = \frac{12}{5}\frac{P'}{b}\frac{1+\nu}{E}\frac{1}{t}\{4\log\frac{h_o+tx}{h_o} - 3\frac{2h_o+5tx}{2(h_o+tx)^2}tx\}\cos^2\alpha \qquad (14a)$$

which, together with Eqs. (12) and (12a), yields:

$$\frac{\delta_{shear1}}{\delta_{bend1}} = \frac{\frac{12}{5}\frac{P}{b}\frac{1+\nu}{E}\cdot\frac{1}{t}\{4\log\frac{h_o+tx}{h_o} - 3\frac{2h_o+5tx}{2(h_o+tx)^2}tx\}}{12\frac{P}{bE}\frac{1}{t^3}\{\log\frac{h_o+tx}{h_o} - \frac{2h_o+3tx}{2(h_o+tx)^2}tx\}} = \frac{\delta'_{shear1}}{\delta'_{bend1}}$$

or

$$\frac{\delta_{shear1}}{\delta_{bend1}} = \frac{\frac{1+\nu}{5}\{4\log\frac{h_o+tx}{h_o} - 3\frac{2h_o+5tx}{2(h_o+tx)^2}tx\}}{\frac{1}{t^2}\{\log\frac{h_o+tx}{h_o} - \frac{2h_o+3tx}{2(h_o+tx)^2}tx\}} = \frac{\delta'_{shear1}}{\delta'_{bend1}} \qquad (15)$$

However, at the limit as $t \to 0$, which is equivalent to a straight cantilever *of uniform cross-section, one gets*

$$\lim_{t\to 0}\delta_{shear1} = \frac{12}{5}\frac{P}{b}\frac{1+\nu}{E}\lim_{t\to 0}\{\frac{4\log\frac{h_o+tx}{h_o}}{t} - 3\frac{2h_o+5tx}{2(h_{oy}tx)^2}x\}$$

where

$$\lim_{t\to 0}\{\frac{\log\frac{h_o+tx}{h_o}}{t}\} = \lim_{t\to 0}\{\frac{\frac{d}{dt}\log\frac{h_o+tx}{h_o}}{\frac{d}{dt}(t)}\} = \lim_{t\to 0}\frac{x}{h_o+tx} = \frac{x}{h_o}$$

and

$$\lim_{t\to 0}\frac{2h_o+5tx}{2(h_o+tx)^2}\cdot x = \frac{x}{h_o}$$

Thus

$$U_{shear1} = \frac{9P^2}{b}\,\frac{4}{15}\,\frac{1+\nu}{E}\int_o^x \frac{h_o - 4h_otx + 4t^2x^2}{(h_o+tx)^3}\,dx$$

for which

$$h_o\int_o^x \frac{dx}{(h_o+tx)^3} = -\frac{h_o^2}{2t}\cdot\frac{1}{(h_o+tx)^2}\Big/_o^x = \frac{h_o^2}{2t}\left[\frac{1}{h_o^2} - \frac{1}{(h_o+tx)^2}\right] = \frac{1}{t}\,\frac{2h_o + tx}{2(h_o+tx)^2}\,tx$$

and

$$-4h_ot\int_o^x \frac{x\cdot dx}{(h_o+tx)^3} = 4h_ot\,\frac{1}{t^2}\left[\frac{1}{h_o+tx} - \frac{h_o}{2(h_o+tx)^2}\right]_o^x = -\frac{1}{t}\,\frac{4t^2x^2}{2(h_o+tx)^2}$$

and

$$4t^2\int_o^x \frac{x^2dx}{(h_o+tx)^3} = \frac{4}{t}\left[\log(h_o+tx) + \frac{2h_o}{h_o+tx} - \frac{h_o^2}{2(h_o+tx)^2}\right]_o^x$$

$$= \frac{4}{t}\left[\log\frac{h_o+tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2}\,tx\right]$$

Thus

$$\int_o^x \frac{h_o^2 - 4h_otx + 4t^2x^2}{(h_o+tx)^3}\,dx = \frac{1}{t}\Big\{4\,\log\frac{h_o + tx}{h_o} +$$

$$\frac{2h_otx + t^2x^2 - 4t^2x^2 - 8h_otx - 12\,t^2x^2}{2(h_o+tx)^2}\Big\}$$

$$= \frac{1}{t}\left\{4\,\log\frac{h_o + tx}{h_o} - 3\,\frac{2h_o + 5tx}{2(h_o+tx)^2}\,tx\right\}$$

from which

$$U_{shear1} = \frac{12}{5}\,\frac{P^2}{b}\,\frac{1+\nu}{E}\,\frac{1}{t}\left\{4\,\log\frac{h_o + tx}{h_o} - 3\,\frac{2h_o + 5tx}{2(h_o+tx)^2}\,tx\right\} = P\delta_{shear1}$$

and thus

$$\delta_{shear1} = \frac{12}{5}\,\frac{P^2}{b}\,\frac{1+\nu}{E}\,\frac{1}{t}\left\{4\,\log\frac{h_o + tx}{h_o} - 3\,\frac{2h_o + 5tx}{2(h_o+tx)^2}\,tx\right\} \qquad (14)$$

If, however, in the derivation of Eq. (12) one uses $P'\cdot\cos\alpha$ for P and is to determine the displacement δ' in the x'-y' coordinate system, one will get the following:

$$\delta'_{bend1} = 12\,\frac{P'}{bE}\,\frac{1}{t^3}\,\{\log\frac{h_o+tx}{h_o} - \frac{2h_o+3tx}{2(h_o+tx)^2}\cdot tx\}\cdot\cos^2\alpha \tag{12a}$$

The local strain energy due to that component of shear stress resulting from the gradient in normal bend stresses and acting on planes normal to the plane of symmetry and in the axial direction, $\sigma_{yx,m}$ (or its equivalent acting on planes parallel to the plane of symmetry and in the direction normal to the plane of symmetry, $\sigma_{xy,m}$) u_{shear1}, is

$$u_{shear1} = \sigma_{yx,m}\epsilon_{yx,m} = 2\,\frac{1+\nu}{E}\,\sigma^2_{yx,m} = 2\cdot\frac{9P^2}{4b^2}\cdot\frac{1+\nu}{E}\,\frac{(h_o-2tx)^2}{(h_o+tx)^4}\,[1-(2\frac{y}{h_o+tx})^2]^2$$

$$= \frac{9P^2}{2b^2}\cdot\frac{1+\nu}{E}\,(h_o^2-4h_otx+4t^2x^2)[(\frac{1}{(h_o+tx)^4} - \frac{8y^2}{(h_o+tx)^6} + \frac{16y^4}{(h_o+tx)^8}] \tag{13}$$

for which the total shear strain energy, U_{shear1}, over the entire volume becomes:

$$U_{shear1} = \frac{9P^2}{2b^2}\,\frac{1+\nu}{E}\cdot b\cdot 2\int_o^x\int_o^{(h_o+tx)/2}(h_o^2-4h_otx+4t^2x^2)\cdot$$

$$[\frac{1}{(h_o+tx)^4} - \frac{8y^2}{(h_o+tx)^6} + \frac{16y^4}{(h_o+tx)^8}]\cdot dy\cdot dx$$

for which

$$\int_o^{(h_o+tx)/2}[\frac{1}{(h_o+tx)^4} - \frac{8y^2}{(h_o+tx)^6} + \frac{16y^4}{(h_o+tx)^8}]dy =$$

$$[\frac{y}{(h_o+tx)^4} - \frac{8y^3}{3(h_o+tx)^6} + \frac{16y^5}{5(h_o+tx)^8}]_o^{(h_o+tx)/2} = \frac{4}{15}\,\frac{1}{(h_o+tx)^3}$$

Deflection Due to Bending Forces Only

The local strain energy due to normal stresses, u_{bend1}, is

$$u_{bend1} = \sigma_{xx,m} \cdot \epsilon_{xx,m} = \frac{\sigma^2_{xx,m}}{E} = 144 \frac{P^2}{b^2E} \frac{x^2y^2}{(h_o+tx)^6} = 144 \frac{P^2}{b^2E} \frac{x^2y^2}{(h_o+tx)^6} \cdot \cos^2 \alpha$$

Thus, the total bending strain energy, U_{bend1}, over the entire volume of the (deformed) beam becomes:

$$U_{bend1} = 2b\int_o^x \int_o^{(h_o+tx)/2} u_{bend} \cdot dy \cdot dx = 288 \frac{P^2}{bE} \int_o^x \int_o^{(h_o+tx)/2} \frac{x^2y^2}{(h_o+tx)^6} \cdot dy \cdot dx$$

where

$$\int_o^{(h_o+tx)/2} \frac{x^2y^2}{(h_o+tx)^6} dy = \frac{1}{3} \frac{x^2}{(h_o+tx)^6} y^3 \Big/_o^{(h_o+tx)/2} = \frac{1}{24} \frac{x^2}{(h_o+tx)^3}$$

for which

$$\int_o^x \frac{x^2}{(h_o+tx)^3} \cdot dx = \frac{1}{t^3} [\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)} tx]$$

or

$$U_{bend1} = \frac{288}{24} \frac{P^2}{bE} \frac{1}{t^3} \{\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx\}$$

$$= 12 \frac{P^2}{bE} \frac{1}{t^3} \{\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx\} = P\delta_{bend1}$$

from which

$$\delta_{bend1} = 12 \frac{P}{bE} \frac{1}{t^3} \{\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx\} \qquad (12)$$

which is the same as y_o arrived at in Eq. (7) through the double integration of the bending moment method, and as such

$$\lim_{t \to 0} \delta_{bend1} = -4 \frac{P}{bE}\left(\frac{x}{h_o}\right)^3$$

as in Eq. (9).

$$= \lim_{t\to 0} \frac{\dfrac{x}{h_o + tx} - \dfrac{2h_o^2x + 8h_otx^2 + 6t^2x^3 - 4h_otx^2 - 6t^2x^3}{2(h_o+tx)^3}}{3t^2}$$

$$= \lim_{t\to 0} \frac{\dfrac{2h_o^2x + 4h_otx^2 + 2t^2x^3 - 2h_o^2x - 8h_otx^2 - 6t^2x^3 + 4h_otx^2 + 6t^2x^3}{2(h_o+tx)^3}}{3t^2}$$

$$= \lim_{t\to 0} \frac{\dfrac{2t^2x^3}{2(h_o+tx)^3}}{3t^2} = \lim_{t\to 0} \frac{x^3}{3(h_o+tx)^3} = \frac{1}{3}\left(\frac{x}{h_o}\right)^3$$

to which one may add that

$$\lim_{t\to 0}\left\{\frac{1}{t}\left[\log\frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2}\cdot tx\right]\right\} = \lim_{t\to 0} \frac{\log\dfrac{h_o + tx}{h_o}}{t} - \lim_{t\to 0} \frac{2h_o + 3tx}{2(h_o+tx)^2}\cdot x$$

where

$$\lim_{t\to 0} \frac{\log\dfrac{h_o + tx}{h_o}}{t} = \lim \frac{x}{h_o + tx} = \frac{x}{h_o}$$

and

$$\lim_{t\to 0} \frac{2h_o + 3tx}{2(h_o+tx)^2}\, x = \frac{x}{h_o}$$

Thus

$$\lim_{t\to 0}\left\{\frac{1}{t}\left[\log\frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2}\cdot tx\right]\right\} = 0$$

Therefore

$$\lim_{t\to 0} \delta_{bend3} = 12\,\frac{P}{bE}\,\frac{1}{3}\left(\frac{x}{h_o}\right)^3 = 4\,\frac{P}{bE}\left(\frac{x}{h_o}\right)^3$$

which, except for the sign, is the same as that arrived at in Eq. (9), and which concurs with the value given in handbooks (ref 2) for a straight cantilever of a uniform cross-section.

Even though the shear stress σ_{yx} approaches zero at the specimen's boundaries, $y = \pm (h_o+tx)/2$, and thus it is not in violation on the boundaries, it is assumed here that, as with the normal stresses, σ_{yx} is the cartesian component of a radial shear stress, $\sigma_{\theta r}$, which converges at (or radiates from) the same cylindrical coordinate origin at $L = -(h_o/t + x)$ as before.

$$\sigma_{\theta r,m} = \sqrt{1 + \left(\frac{ty}{h_o+tx}\right)^2}$$

or

$$\sigma^2_{\theta r,m} = \left[1 + \frac{t^2y^2}{(h_o+tx)^2}\right] \cdot \sigma^2_{yx,m}$$

Thus, since

$$\sigma_{yx,m} = \frac{3P}{2b} \cdot \frac{h_o - 2tx}{(h_o+tx)^2} \left[1 - \left(2 \frac{y}{h_o+tx}\right)^2\right]$$

$$\sigma_{\theta r,m} = \frac{3P}{2b} \frac{h_o - 2tx}{(h_o+tx)^2} \cdot \left[1 - \left(2 \frac{y}{h_o+tx}\right)\right]^2 \cdot \sqrt{1 + \left(\frac{ty}{h_o+tx}\right)^2}$$

Thus, the local shear strain energy due to the bending forces, u_{shear3} is

$$u_{shear3} = \sigma_{\theta r,m}\varepsilon_{\theta r,m} = 2 \frac{1+\nu}{E} \sigma^2_{\theta r,m} = \frac{9P^2}{4b^2} \frac{1+\nu}{E} (h_o^2 - 4h_o tx + 4t^2x^2) \cdot$$

$$\left[\frac{1}{(h_o+tx)^4} - \frac{8y^2}{(h_o+tx)^6} + \frac{16y^4}{(h_o+tx)^8}\right] \cdot \left[1 + \frac{t^2y^2}{(h_o+tx)^2}\right]$$

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[2]Raymond J. Roark and Warren C. Young, Formulas for Stress and Strain, Fifth Edition, McGraw-Hill, NY, 1975, p. 98.

from which the total shear energy due to bending forces only over the entire beam's volume

$$U_{shear3} = 4 \int_o^x \int_o^{(h_o+tx)/2} \int_o^{b/2} u_{shear3} \cdot dz \cdot dy \cdot dx$$

$$= \frac{9P^2}{2b} \frac{1+\nu}{E} \int_o^x \int_o^{(h_o+tx)/2} (h_o{}^2-4h_otx+4t^2x^2) \cdot [\frac{1}{(h_o+tx)^4} - \frac{8y^2}{(h_o+tx)^6} + \frac{16y^4}{(h_o+tx)^8}] \cdot$$

$$[1 + \frac{t^2y^2}{(h_o+tx)^2}]dydx$$

where

$$\int_o^{(h_o+tx)/2} (h_o{}^2-4h_otx+4t^2x^2) \cdot [\frac{1}{(h_o+tx)^4} - \frac{8y^2}{(h_o+tx)^6} + \frac{16y^4}{(h_o+tx)^8}] \cdot [1 + \frac{t^2y^2}{(h_o+tx)^2}]dy$$

$$= (h_o{}^2-4h_otx+4t^2x^2) \cdot [\frac{y}{(h_o+tx)^4} - \frac{(8-t^2)y^3}{3(h_o+tx)^6} + \frac{8(2-t^2)y^5}{5(h_o+tx)^8} + \frac{16t^2y^7}{7(h_o+tx)^{10}}]_o^{(h_o+tx)/2}$$

$$= \frac{h_o{}^2 - 4h_otx + 4t^2x^2}{(h_o+tx)^3} [\frac{1}{2} - \frac{8-t^2}{24} + \frac{2-t^2}{20} + \frac{t^2}{56}]$$

$$= \frac{h_o{}^2 - 4h_otx + 4t^2x^2}{(h_o+tx)^3} [(\frac{1}{2} - \frac{1}{3} + \frac{1}{10}) + (\frac{1}{24} - \frac{1}{20} + \frac{1}{56})t^2] =$$

$$(\frac{4}{15} + \frac{t^2}{105}) \frac{h_o - 4h_otx + 4t^2x^2}{(h_o+tx)^3}$$

$$= \frac{1}{15} (4 + \frac{t^2}{7}) \frac{h_o - 4h_otx + 4t^2x^2}{(h_o+tx)^3}$$

Thus,

$$U_{shear3} = \frac{3}{10} (4 + \frac{t^2}{7}) \frac{P^2}{b} \frac{1+\nu}{E} \int_o^x \frac{h_o - 4h_otx + 4t^2x^2}{(h_o+tx)^3} dx$$

where, as in the derivation of Eq. (14)

$$\int_0^x \frac{h_o^2 - 4h_o tx + 4t^2x^2}{(h_o+tx)^3} dx = \frac{1}{t} \{4 \cdot \log \frac{h_o+tx}{h_o} - 3 \frac{2h_o + 5tx}{2(h_o+tx)^2} tx\}$$

Thus

$$U_{shear3} = \frac{3}{10} (\frac{4}{t} + \frac{t}{7}) \frac{P^2}{b} \frac{1+\nu}{E} \{4 \cdot \log \frac{h_o + tx}{h_o} - 3 \frac{2h_o + 5tx}{2(h_o+tx)^2} tx\} = P\delta_{shear3}$$

Hence

$$\delta_{shear3} = \frac{3}{10} (\frac{4}{t} + \frac{t}{7}) \frac{P}{b} \frac{1+\nu}{E} \{4 \cdot \log \frac{h_o + tx}{h_o} - 3 \frac{2h_o + 5tx}{2(h_o+tx)^2} tx\} \qquad (27)$$

and as before,

$$\delta'_{shear3} = \frac{3}{10} (\frac{4}{t} + \frac{t}{7}) \frac{P}{b} \frac{1+\nu}{E} \{4 \cdot \log \frac{h_o + tx}{h_o} - 3 \frac{2h_o + 5tx}{2(h_o+tx)^2} tx\}\cos^2 \alpha \qquad (27a)$$

Since

$$\lim_{t \to 0} \{t[4 \cdot \log \frac{h_o + tx}{h_o} - 3 \frac{2h_o + 5tx}{2(h_o+tx)^2} tx]\} = 0$$

as in the derivation of Eq. (16) above

$$\lim_{t \to 0} \delta_{shear3} = \frac{12}{5} \frac{P}{b} \frac{1+\nu}{E} (\frac{x}{h_o})$$

as it should be. Also,

$$\frac{\delta_{shear3}}{\delta_{bend3}} = \frac{\frac{3}{10} (\frac{4}{t} + \frac{t}{7}) \frac{P}{b} \frac{1+\nu}{E} \{4 \cdot \log \frac{h_o + tx}{h_o} - 3 \frac{2h_o + 5tx}{2(h_o+tx)^2} tx\}}{(\frac{12}{t^3} + \frac{9}{5} \frac{1}{t}) \frac{P}{bE} \{\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx\}}$$

or

$$\frac{\delta_{shear3}}{\delta_{bend3}} = \frac{\frac{3}{10} (\frac{4}{t} + \frac{t}{7})(1+\nu) \{4 \cdot \log \frac{h_o + tx}{h_o} - 3 \frac{2h_o + 5tx}{2(h_o+tx)^2} tx\}}{(\frac{12}{t^3} + \frac{9}{5} \frac{1}{t}) \{\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx\}} \qquad (28)$$

Deflection Due to Combined Bending and Compressive Forces

The same process of considering the converging stresses used in the above calculations of the gap-opening due to the bending forces can be used for the combined bending and compressive forces.

As in Eq. (19) before, the Cartesian component of the combined forces

$$\sigma_{xx} = \sigma_{xx,m} + \sigma_{xx,p} = \frac{P}{2b}\left\{24\,\frac{xy}{(h_o+tx)^2} - t\right\}\frac{1}{h_o + tx}$$

Thus, as before, the full normal stress will be

$$\sigma_{rr,m} = \frac{P}{2b}\sqrt{1 + \left(\frac{ty}{h_o+tx}\right)^2}\cdot\left\{24\,\frac{xy}{(h_o+tx)^2} - t\right\}\cdot\frac{1}{h_o + tx}$$

and the local strain energy due to normal stresses, u_{bend4}, will be

$$u_{bend4} = \sigma_{rr}\varepsilon_{rr} = \frac{\sigma^2_{rr}}{E} = \frac{P^2}{4b^2}\left[1 + \frac{t^2y^2}{(h_o+tx)^2}\right]\cdot$$

$$\left\{576\,\frac{x^2y^2}{(h_o+tx)^6} - 48\,\frac{txy}{(h_o+tx)^4} + \frac{t^2}{(h_o+tx)^2}\right\}$$

However, due to the beam's symmetry, when integrating throughout the entire beam's volume, the term

$$48\,\frac{txy}{(h_o+tx)^4}$$

from both sides of the plane of symmetry cancels itself, and one gets for the total bending strain energy, U_{bend4}, the following:

$$U_{bend4} = \int_o^x \int_o^{h_o+tx/2} \int_o^{b/2} u_{bend4}\cdot dz\cdot dy\cdot dx = \frac{P^2}{2bE}\int_o^x \int_o^{(h_o+tx)/2} \left[1 + \frac{t^2y^2}{(h_o+tx)^2}\right]\cdot$$

$$\left\{576\,\frac{x^2y^2}{(h_o+tx)^6} + \frac{t^2}{(h_o+tx)^2}\right\}\cdot dy\cdot dx$$

where

$$\int_o^{(h_o+tx)/2} [1 + \frac{t^2y^2}{(h_o+tx)^2}] \cdot \{576 \frac{x^2y^2}{(h_o+tx)^6} + \frac{t^2}{(h_o+tx)^2}\}dy$$

$$= [\frac{t^2y}{(h_o+tx)^2} + (576 \frac{x^2}{(h_o+tx)^6} + \frac{t^4}{(h_o+tx)^4}) \frac{y^3}{3} + 576 \frac{t^2x^2y^5}{5(h_o+tx)^8}]_o^{(h_o+tx)/2}$$

$$= [(24 + \frac{18}{5} t^2) \frac{x^2}{(h_o+tx)^3} + (\frac{1}{24} + \frac{t^2}{2}) \cdot \frac{t^2}{h_o + tx}]$$

Thus

$$U_{bend4} = \frac{P^2}{2bE} \int_o^x [(24 + \frac{18}{5} t^2) \cdot \frac{x^2}{(h_o+tx)^3} + (\frac{1}{24} + \frac{t^2}{2}) \cdot \frac{t^2}{h_o + tx}] dx$$

where

$$\int_o^x \frac{x^2}{(h_o+tx)^3} dx = \frac{1}{t^3} [\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx]$$

and

$$\int_o^x \frac{dx}{h_o + tx} = \frac{1}{t} \log \frac{h_o + tx}{h_o}$$

as has been shown before. Thus,

$$U_{bend4} = \frac{P^2}{2bE} \{(\frac{24}{t^3} + \frac{18}{5} \frac{1}{t})[\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2} tx] + (\frac{1}{24} t + \frac{t^3}{2})\log \frac{h_o + tx}{h_o}\}$$

$$= P\delta_{bend4}$$

Thus

$$\delta_{bend4} = \frac{P}{2bE} \{(\frac{24}{t^3} + \frac{18}{5} \frac{1}{t})[\log \frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^3} tx]$$

$$+ (\frac{1}{24} t + \frac{t^3}{2})\log \frac{h_o + tx}{h_o}\} \qquad (29)$$

and

$$\delta'_{bend4} = \frac{P}{2bE}\left\{\left(\frac{24}{t^3} + \frac{18}{5}\cdot\frac{1}{t}\right)\left[\log\frac{h_o + tx}{h_o} - \frac{2h_o + 3tx}{2(h_o+tx)^2}tx\right]\right.$$

$$\left. + \left(\frac{1}{24}t + \frac{t^3}{2}\right)\log\frac{h_o + tx}{h_o}\right\}\cos^2\alpha \qquad (29a)$$

Since

$$\lim_{t \to 0}\left\{\left(\frac{1}{24}t + \frac{t^3}{2}\right)\log\frac{h_o + tx}{h_o}\right\} = 0$$

as before

$$\lim_{t \to 0}\delta_{bend4} = 4\frac{P}{bE}\left(\frac{x}{h_o}\right)^3$$

Incorporating the bending and the compressive components of the applied force into the computation of the resultant radial shear stresses, $\sigma_{\theta r}$, yields the following:

$$\sigma_{\theta r} = \sigma_{\theta r,m} + \sigma_{\theta r,p} = \frac{P}{2b}\left\{3\frac{h_o - 2tx}{(h_o+tx)^2}\left[1 - \left(2\frac{y}{h_o+tx}\right)^2\right] - \frac{t^2}{2(h_o+tx)}\cdot\right.$$

$$\left.\left[1 - 2\frac{|y|}{h_o + tx}\right]\right\}\sqrt{1 + \left(\frac{ty}{h_o+tx}\right)^2}$$

Thus the local shear strain energy resulting from the radial shear stresses due to the combined bending and compressive forces is

$$u_{shear4} = \sigma_{\theta r}\varepsilon_{\theta r} = 2\frac{1+\nu}{E}\sigma^2_{\theta r}$$

$$\frac{P^2}{2b^2}\frac{1+\nu}{E}\left\{9\frac{h_o^2 - 4h_otx + 4t^2x^2}{(h_o+tx)^4}\left[1 - 2\left(2\frac{y}{h_o+tx}\right)^2 + \left(2\frac{y}{h_o+tx}\right)^4\right]\right.$$

$$-3\frac{h_o - 2tx}{(h_o+tx)^3}t^2\left[1 - 2\frac{|y|}{h_o + tx} - \left(2\frac{y}{h_o+tx}\right)^2 + \left(2\frac{|y|}{h_o+tx}\right)^3\right]$$

$$\left. + \frac{t^4}{4(h_o+tx)^2}\left[1 - 4\frac{|y|}{h_o+tx} + \left(2\frac{y}{h_o+tx}\right)^2\right]\right\}\cdot\left[1 + \frac{t^2y^2}{(h_o+tx)^2}\right]$$

However, when integrating over the entire volume one can omit the term

$$3\,\frac{h_o - 2tx}{(h_o+tx)^3}\,t^2\,[1 - 2\,\frac{|y|}{h_o + tx} - (2\,\frac{y}{h_o+tx})^2 + (2\,\frac{|y|}{h_o+tx})^3]$$

since it cancels itself on both sides of the plane of symmetry. Therefore

$$U_{shear4} = 4\int_o^x \int_o^{(h_o+tx)/2} \int_o^{b/2} u_{shear4}\cdot dz\cdot dy\cdot dx$$

$$= \frac{P^2}{b}\,\frac{1+\nu}{E}\int_o^x \int_o^{(h_o+tx)/2} \{9\,\frac{h_o^2 - 4h_o tx + 4t^2x^2}{(h_o+tx)^4}\,[1 - 2(2\,\frac{y}{h_o+tx})^2 + (2\,\frac{y}{h_o+tx})^4]$$

$$+ \frac{t^4}{4(h_o+tx)^2}\,[1 - 4\,\frac{|y|}{h_o+tx} + (2\,\frac{y}{h_o+tx})^2]\}\cdot[1 + \frac{t^2y^2}{(h_o+tx)^2}]\cdot dy\cdot dx$$

where

$$\int_o^{(h_o+tx)/2} \{9\,\frac{h_o^2 - 4h_o tx + 4t^2x^2}{(h_o+tx)^4}\cdot[1 - 2(2\,\frac{y}{h_o+tx})^2 + (2\,\frac{y}{h_o+tx})^4]$$

$$+ \frac{t^4}{4(h_o+tx)^2}\cdot[1 - 4\,\frac{|y|}{h_o+tx} + (2\,\frac{y}{h_o+tx})^2]\}\cdot[1 + \frac{t^2y^2}{(h_o+tx)^2}]\cdot dy$$

$$= \{9\,\frac{h_o^2 - 4h_o tx + 4t^2x^2}{(h_o+tx)^4}\cdot[y - \frac{8-t^2}{(h_o+tx)^2}\cdot\frac{y^3}{3} + \frac{16 - 8t^2}{(h_o+tx)^4}\cdot\frac{y^5}{5} + \frac{16t^2}{(h_o+tx)^6}\cdot\frac{y^7}{7}]$$

$$+ \frac{t^4}{4(h_o+tx)^2}[y - \frac{4}{h_o + tx}\cdot\frac{y^2}{2} + \frac{4+t^2}{(h_o+tx)^2}\cdot\frac{y^3}{3} - \frac{4t^2}{(h_o+tx)^3}\cdot\frac{y^4}{4} + \frac{4t^2}{(h_o+tx)^4}\cdot\frac{y^5}{5}]\}_o^{(h_o+tx)/2}$$

$$= 9\,\frac{h_o^2 - 4h_o tx + 4t^2x^2}{(h_o+tx)^3}\,[\frac{1}{2} - \frac{8-t^2}{24} + \frac{2-t^2}{20} + \frac{t^2}{56}] + \frac{t^4}{4(h_o+tx)}\cdot$$

$$[\frac{1}{2} - \frac{1}{2} + \frac{4+t^2}{24} - \frac{t^2}{16} + \frac{t^2}{20}]$$

$$= \frac{3}{5}\left(4 + \frac{t^2}{7}\right) \frac{h_o - 4h_o tx + 4t^2x^2}{(h_o+tx)^3} + \left(\frac{1}{6} + \frac{7}{240} t^2\right) \frac{t^4}{4(h_o+tx)}$$

for which

$$\int_o^x \frac{h_o - 4h_o tx + 4t^2x^2}{(h_o+tx)^3} dx = \frac{1}{t} \{4 \cdot \log \frac{h_o + tx}{h_o} - 3 \frac{2h_o + 5tx}{2(h_o+tx)^2} tx\}$$

and

$$\int_o^x \frac{dx}{h_o + tx} = \frac{1}{t} \log \frac{h_o + tx}{h_o}$$

Thus

$$U_{shear4} = \frac{P^2(1+\nu)}{bE} \{\frac{3}{5} \cdot \left(\frac{4}{t} + \frac{t}{7}\right) \cdot [4 \cdot \log \frac{h_o + tx}{h_o} - 3 \frac{2h_o + 5tx}{2(h_o+tx)^2} tx]$$

$$+ \left(\frac{t^3}{6} + \frac{7}{240} t^5\right) \cdot \log \frac{h_o + tx}{h_o}\} = P\delta_{shear4}$$

from which

$$\delta_{shear4} = \frac{P}{b} \frac{1+\nu}{E} \{\frac{3}{5} \cdot \left(\frac{4}{t} + \frac{t}{7}\right) \cdot [4 \cdot \log \frac{h_o + tx}{h_o} - 3 \frac{2h_o + 5tx}{2(h_o+tx)^2} tx]$$

$$+ \left(\frac{t^3}{6} + \frac{7}{240} t^5\right) \cdot \log \frac{h_o + tx}{h_o}\} \tag{30}$$

and similarly

$$\delta'_{shear4} = \frac{P'}{b} \frac{1+\nu}{E} \{\frac{3}{5} \cdot \left(\frac{4}{t} + \frac{t}{7}\right) \cdot [4 \cdot \log \frac{h_o + tx}{h_o} - 3 \frac{2h_o + 5tx}{2(h_o+tx)^2} tx]$$

$$+ \left(\frac{t^3}{6} + \frac{7}{240} t^5\right) \cdot \log \frac{h_o + tx}{h_o}\} \cos^2 \alpha \tag{30a}$$

since

$$\lim_{t \to 0} \left(\frac{t^3}{6} + \frac{7}{240} t^5\right) \log \frac{h_o + tx}{h_o} = 0$$

as before

$$\lim_{t \to 0} \delta_{shear4} = \frac{3}{5} (1+\nu)\left(\frac{h_o}{x}\right)^2$$

and

$$\frac{\delta_{shear4}}{\delta_{bend4}} =$$

$$\frac{(1+\nu)\{\frac{3}{5}\cdot(\frac{4}{t}+\frac{t}{7})\cdot[4\cdot\log\frac{h_o+tx}{h_o} - 3\,\frac{2h_o+5tx}{2(h_o+tx)^2}\,tx] + (\frac{1}{6}+\frac{7}{240}\,t^2)t^3\cdot\log\frac{h_o+tx}{h_o}}{\frac{1}{2}\,\{(\frac{24}{t^3}+\frac{18}{5}\cdot\frac{1}{t})\cdot[\log\frac{h_o+tx}{h_o} - \frac{2h_o+3tx}{2(h_o+tx)^2}\,tx] + (\frac{1}{24}+\frac{t^2}{2})t\cdot\log\frac{h_o+tx}{h_o}} \tag{31}$$

SUMMARY

This report provides equations for the determination of that part of the gap opening in fracture toughness specimens, which is due to the deflection of the cantilever portion of the specimen. The method used is based on beam theory and it covers elastic deformation only. Four different field combinations were considered.

1. Where only the stress components which are parallel (or normal) to the cantilever beam's plane of symmetry are being considered.

2. Where the stresses are assumed to be parallel to the beam's outer surfaces, at these surfaces, and where they gradually rotate in between.

Each of the above stress fields was considered as the result of

a. the bending component of the applied force only, and

b. the combined bending and compressive components of the applied force.

In all of the above cases, the contributions of bending stresses and shear stresses were computed. Table I is a computer printout for the computed half gap opening for varying angles of specimen taper and for varying crack length to beam's cross-sectional height ratios. The gap openings to be anticipated by each of the above assumed stress fields are compared.

REFERENCES

1. Boaz Avitzur, "Retained Deflection in Circular and Concentrically Hollowed Beams After Local Removal," to be published.
2. Raymond J. Roark and Warren C. Young, Formulas for Stress and Strain, Fifth Edition, McGraw-Hill, NY, 1975, p. 98.
3. A. C. Ugural and S. K. Fenster, Advanced Strength and Applied Elasticity, American Elsevier, NY, 1981, pp. 146-148.

TABLE I. GAP OPENING (HALF) IN TAPERED DOUBLE CANTILEVER BEAMS

Following is a chart of the computed half gap opening in tapered double cantilever beams whose material properties are:

Modulus of Elasticity	=	0.30000E+08 pounds per inch square
Yield Strength	=	0.16000E+06 pounds per inch square
Poisson's Ratio	=	0.30000
Beam's Width	=	1.00000 inches
Beam's Height at Load	=	0.50000 inches
Starting Crack Length	=	0.50000 inches

Crack Length Over Height at Load
x/h_o Prime = 1.50000

Taper $t = 2 \cdot \tan \alpha$	Relative Crack Length x/h_o	1. Untapered 2. Tapered Load	1. Untapered 2. Mode 1 3. Mode 2 4. Mode 3 5. Mode 4 δ_{bend}	δ_{shear}	$\delta_{total} = \delta_{bend} + \delta_{shear}$	$\delta_{shear} / \delta_{bend}$
0.1CCC0	1.47132	0.88889E+04	0.40000E-02	0.13867E-02	0.538€7E-C2	0.34667E+00
		0.11833E+05	0.36757E-02	C.11079E-02	0.47835E-02	0.30141E+00
			0.36770E-02	0.11079E-02	0.47849E-02	0.30130E+00
			0.36812E-02	0.11083E-02	0.47894E-02	0.30106E+C0
			0.36813E-02	0.11083E-02	0.47896E-02	0.30106E+00
0.2CC00	1.43564	0.88889E+04	0.400ააE-02	0.13867E-02	0.53867E-02	0.34667E+00
		C.15085E+C5	0.33268E-02	C.90468E-03	0.42315E-02	0.27153E+C0
			0.33331E-02	0.90474E-03	0.42379E-02	0.27144E+00
			0.33468E-02	0.90598E-03	0.42528E-02	0.27070E+00
			0.33476E-02	0.90620E-03	0.42538E-02	0.27070E+00
0.3CCC0	1.39364	0.88889E+04	0.40000E-02	0.13867E-02	0.538€7E-C2	0.34667E+00
		0.18555E+05	0.29636E-02	0.76505E-03	0.37286E-02	0.25815E+00
			0.29794E-02	0.76536E-03	0.37448E-02	0.25688E+00
			0.30036E-02	0.76751E-03	0.37711E-02	0.25553E+C0
			0.30063E-02	0.76877E-03	0.37751E-02	0.25572E+00

TABLE I. GAP OPENING (HALF) IN TAPERED DOUBLE CANTILEVER BEAMS (CONT'D)

Taper $t = 2 \cdot \tan \alpha$	Relative Crack Length x/h_o	Load (1. Untapered, 2. Tapered)	δ_{bend} (1. Untapered, 2. Mode 1, 3. Mode 2, 4. Mode 3, 5. Mode 4)	δ_{shear}	$\delta_{total} = \delta_{bend} + \delta_{shear}$	$\frac{\delta_{shear}}{\delta_{bend}}$
0.40000	1.34615	0.88889E+04	0.40000E-02	0.13867E-02	0.53867E-02	0.34667E+00
		0.22144E+05	0.25994E-02	0.67474E-03	0.32741E-02	0.25958E+00
			0.26299E-02	0.67580E-03	0.33057E-02	0.25696E+00
			0.26618E-02	0.67859E-03	0.33403E-02	0.25494E+00
			0.26692E-02	0.68295E-03	0.33521E-02	0.25586E+00
0.50000	1.29412	0.88889E+04	0.40000E-02	0.13867E-02	0.53867E-02	0.34667E+00
		0.25750E+05	0.22463E-02	0.61941E-03	0.28657E-02	0.27574E+00
			0.22967E-02	0.62214E-03	0.29189E-02	0.27088E+00
			0.23306E-02	0.62494E-03	0.29555E-02	0.26815E+00
			0.23474E-02	0.63633E-03	0.29837E-02	0.27108E+00
0.60000	1.23853	0.88889E+04	0.40000E-02	0.13867E-02	0.53867E-02	0.34667E+00
		0.29271E+05	0.19140E-02	0.58605E-03	0.25000E-02	0.30619E+00
			0.19886E-02	0.59187E-03	0.25805E-02	0.29763E+00
			0.20174E-02	0.59358E-03	0.26109E-02	0.29424E+00
			0.20504E-02	0.61833E-03	0.26688E-02	0.30156E+00
0.70000	1.18040	0.88889E+04	0.40000E-02	0.13867E-02	0.53867E-02	0.34667E+00
		0.32616E+05	0.16093E-02	0.56402E-03	0.21733E-02	0.35048E+00
			0.17113E-02	0.57486E-03	0.22862E-02	0.33591E+00
			0.17275E-02	0.57389E-03	0.23014E-02	0.33220E+00
			0.17861E-02	0.62096E-03	0.24070E-02	0.34767E+00
0.80000	1.12069	0.88889E+04	0.40000E-02	0.13867E-02	0.53867E-02	0.34667E+00
		0.35705E+05	0.13364E-02	0.54558E-03	0.18820E-02	0.40825E+00
			0.14677E-02	0.56379E-03	0.20315E-02	0.38413E+00
			0.14647E-02	0.55805E-03	0.20227E-02	0.38101E+00
			0.15597E-02	0.63906E-03	0.21987E-02	0.40974E+00

TABLE I. GAP OPENING (HALF) IN TAPERED DOUBLE CANTILEVER BEAMS (CONT'D)

Taper	Relative Crack Length	1. Untapered 2. Tapered	1. Untapered 2. Mode 1 3. Mode 2 4. Mode 3 5. Mode 4			
$t = 2 \cdot \tan \alpha$	x/h_o	Load	δ_{bend}	δ_{shear}	$\delta_{total} = \delta_{bend} + \delta_{shear}$	$\frac{\delta_{shear}}{\delta_{bend}}$
0.90000	1.06029	0.88889E+04	0.40000E-02	0.13867E-02	0.53867E-02	0.34667E+00
		0.38477E+05	0.10973E-02	0.52573E-03	0.16230E-02	0.47918E+00
			0.12580E-02	0.55400E-03	0.18120E-02	0.44036E+00
			0.12306E-02	0.54099E-03	0.17716E-02	0.43963E+00
			0.13742E-02	0.66987E-03	0.20441E-02	0.48746E+00
1.00000	1.00000	0.88889E+04	0.40000E-02	0.13867E-02	0.53867E-02	0.34667E+00
		0.40889E+05	0.89165E-03	0.50208E-03	0.13937E-02	0.56309E+00
			0.10806E-02	0.54302E-03	0.16236E-02	0.50252E+00
			0.10254E-02	0.52001E-03	0.15454E-02	0.50713E+00
			0.12301E-02	0.71242E-03	0.19425E-02	0.57916E+00
1.10000	0.94050	0.88889E+04	0.40000E-02	0.13867E-02	0.53867E-02	0.34667E+00
		0.42910E+05	0.71783E-03	0.47373E-03	0.11916E-02	0.65996E+00
			0.93232E-03	0.52997E-03	0.14623E-02	0.56844E+00
			0.84811E-03	0.49421E-03	0.13423E-02	0.58271E+00
			0.11255E-02	0.76677E-03	0.18923E-02	0.68126E+00
1.20000	0.88235	0.88889E+04	0.40000E-02	0.13867E-02	0.53867E-02	0.34667E+00
		0.44533E+05	0.57307E-03	0.44123E-03	0.10143E-02	0.76994E+00
			0.80953E-03	0.51500E-03	0.13245E-02	0.63619E+00
			0.69685E-03	0.46392E-03	0.11608E-02	0.66574E+00
			0.10571E-02	0.83339E-03	0.18904E-02	0.78841E+00
1.30000	0.82601	0.88889E+04	0.40000E-02	0.13867E-02	0.53867E-02	0.34667E+00
		0.45760E+05	0.45412E-03	0.40573E-03	0.85985E-03	0.89346E+00
			0.70830E-03	0.49881E-03	0.12071E-02	0.70423E+00
			0.56923E-03	0.43022E-03	0.99946E-03	0.75579E+00
			0.10200E-02	0.91263E-03	0.19326E-02	0.89474E+00

TABLE I. GAP OPENING (HALF) IN TAPERED DOUBLE CANTILEVER BEAMS (CONT'D)

Taper $t = 2\cdot\tan\alpha$	Relative Crack Length x/h_o	1. Untapered 2. Tapered Load	1. Untapered 2. Mode 1 3. Mode 2 4. Mode 3 5. Mode 4 δ_{bend}	δ_{shear}	$\delta_{total} = \delta_{bend} + \delta_{shear}$	$\dfrac{\delta_{shear}}{\delta_{bend}}$
1.4CCCC	C.77181	0.89889E+04	0.40000E-02	0.13867E-02	0.53867E-C2	0.34667E+00
		0.46604E+05	0.35750E-03	C.36864E-03	0.72614E-03	0.10312E+01
			0.62484E-03	0.48218E-03	0.11070E-02	C.77168E+C0
			0.46260E-03	0.39445E-03	0.85705E-C3	C.85268E+C0
			0.10089E-02	0.10043E-02	0.20132E-C2	C.99551E+C0
1.5CCCC	0.72CCC	C.98889E+04	0.400CCE-02	0.13867E-02	0.53867E-02	C.34667E+00
		C.47C9CE+C5	C.27981E-03	0.33135E-03	0.61116E-C3	C.11842E+C1
			0.55571E-03	0.46585E-03	0.10216E-C2	C.83830E+C0
			0.37424E-03	0.35798E-03	0.73222E-03	0.95654E+00
			0.1018CE-02	0.11078E-02	C.21258E-C2	0.10882E+01
1.6CCCC	0.67073	C.98889E+04	C.40000E-02	0.13867E-02	0.53867E-02	C.34667E+C0
		0.47246E+05	0.21790E-03	C.29504E-C3	C.51295E-03	0.13540E+01
			0.49795E-03	0.45038E-03	0.94833E-03	0.90446E+00
			C.30158E-03	0.32202E-03	0.62359E-C3	C.10678E+C1
			0.10418E-02	0.12217E-02	0.22635E-C2	0.11726E+01
1.7CCCC	C.62409	0.88889E+04	0.40000E-02	0.13867E-02	0.53867E-02	0.34667E+00
		C.471C6E+C5	0.16895E-03	0.26062E-03	0.42957E-03	C.15426E+C1
			0.44911E-03	0.43606E-03	0.88517E-03	0.97092E+00
			0.24218E-03	0.28752E-03	0.52971E-03	0.11872E+01
			0.10752E-02	0.13441E-02	0.24194E-02	0.12501E+01
1.8CCC0	0.58011	0.88889E+04	0.40000E-02	0.13867E-02	0.53867E-C2	0.34667E+00
		0.46703E+05	0.13049E-03	0.22872E-03	0.35921E-03	0.17528E+01
			0.40723E-03	0.42299E-03	0.83022E-03	0.10387E+01
			0.19390E-03	0.25519E-03	0.44909E-03	0.13160E+C1
			0.11136E-02	0.14729E-02	0.25865E-C2	0.13226E+01

TABLE I. GAP OPENING (HALF) IN TAPERED DOUBLE CANTILEVER BEAMS (CONT'D)

Taper $t = 2 \cdot \tan \alpha$	Relative Crack Length x/h_o	1. Untapered 2. Tapered Load	1. Untapered 2. Mode 1 3. Mode 2 4. Mode 3 5. Mode 4 δ_{bend}	δ_{shear}	$\delta_{total} = \delta_{bend} + \delta_{shear}$	$\frac{\delta_{shear}}{\delta_{bend}}$
1.85999	0.53877	0.88889E+04	0.400CCE-02	0.13867E-02	0.53867E-02	0.34667E+00
		C.46C75E+C5	C.10045E-03	0.19968E-03	0.30013E-03	0.19879E+C1
			0.37075E-03	0.41110E-03	0.78185E-03	0.11088E+01
			0.15484E-03	0.22543E-03	0.38027E-03	0.14559E+01
			0.11531E-02	0.16053E-02	0.27585E-02	0.13922E+01
1.95999	0.5C0C0	0.88889E+04	0.40000E-02	0.13867E-02	0.53867E-C2	0.34667E+00
		0.45255E+05	0.77100E-04	0.17366E-03	0.25076E-03	0.22523E+01
			0.33850E-03	0.40020E-03	0.73871E-C3	0.11823E+C1
			0.12336E-03	0.19846E-03	0.32182E-C3	C.16088E+C1
			0.11907E-02	0.17390E-02	0.29297E-02	0.14604E+01

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